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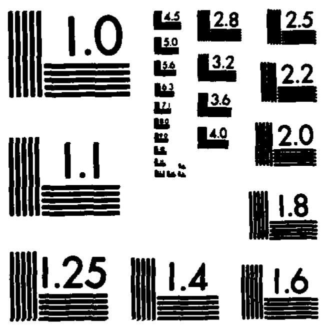

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NON-EXISTENCE OF GLOBAL SOLUTIONS OF $\square u = \frac{\partial}{\partial t} F(u_t)$ IN TWO AND THREE SPACE DIMENSIONS

Fritz John

partial of F (u and t) with respect to t

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ABSTRACT

This paper deals with solutions $u(x_1,\ldots,x_n,t) = u(x,t)$ of nonlinear partial differential equations of the form $\Box u = u_{tt} - \Delta u = F'(u_t)u_{tt}$ for prescribed initial values $u(x,0) = \varepsilon\phi(x)$, $u_t(x,0) = \varepsilon\psi(x)$ of compact support. Here the assumptions $F(0) = F'(0) = 0$, $F'' > 0$, $F' < q < 1$ ensure hyperbolicity of the equation. It is known that for $n > 3$ smooth solutions exist for $x \in \mathbb{R}^n$ and all $t > 0$, provided ε is sufficiently small. It is shown here that no such "global" solutions need to exist for arbitrarily small ε, when $n = 2$ or 3. More precisely, if ϕ and ψ satisfy certain inequalities there exist positive constants A,B such that no classical solution exists for $t > Ae^{B/\varepsilon}$ when $n = 3$ and for $t > A/\varepsilon^2$ when $n = 2$. These upper bounds for the "life span" of u are optimal. For the proof one shows that certain plane integrals of u become larger for large t than is consistent with the value of the total energy derived from the initial data.

AMS (MOS) Subject Classifications: 35L67, 35L70, 73C50

Key Words: singularity formation, blow-up, global existence, life span of solutions

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Solutions of nonlinear hyperbolic partial differential equations often develop singularities spontaneously. Physically this phenomenon corresponds to the formation of shocks in nonlinear waves. One is confronted with the questions: What are the factors contributing to this "blow-up" of solutions? How long does it take for blow-up to develop (i.e. what is the "life span" T of the solution)? What goes on precisely during blow-up? There is no general answer covering the great variety of situations encountered. A critical role certainly is played by the size of the initial disturbance that gives rise to the wave solution, and by the number of dimensions of the space in which the wave propagates. One finds that larger disturbances are more likely to result in shocks, and that, on the other hand, with increased dimension there are more possibilities for the wave to spread out and to decay, thus counteracting the formation of shocks.

This document ~~The present investigation~~ is concerned with a special type of second order nonlinear wave equation, whose behavior can be expected to be typical for a large class of equations occurring in applications, e.g. in the propagation of waves of finite amplitude in elastic materials. Recent results of S. Klainerman show that no blow-up at all occurs (i.e. that $T = \infty$), if the number of space dimensions exceeds 3 and the size ε of the initial disturbance is sufficiently small. Moreover in 3 dimensions T, if not infinite, is extremely large, namely of exponential order in $1/\varepsilon$. The present paper deals with 2 and 3 dimensions. It shows that in 3 dimensions T actually can be finite and of exponential order in $1/\varepsilon$, while in two dimensions (a case studied rarely up to now) T need not exceed the much smaller order $1/\varepsilon^2$. It is known that T cannot possibly be of still smaller order, so that the results given here are optimal.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NON-EXISTENCE OF GLOBAL SOLUTIONS OF $\Box u = \frac{\partial}{\partial t} F(u_t)$ IN TWO AND THREE SPACE DIMENSIONS

Fritz John

This paper deals with solutions $u(x_1,\ldots,x_n,t) = u(x,t)$ of certain nonlinear hyperbolic equations of the form

$$u_{tt} - \sum_{i,k=1}^{n} a_{ik}(u')u_{x_i x_k} = 0 . \tag{1}$$

(Here u' stands for the set of first partial derivatives of u). Equations or systems of equations of type (1) describe the propagation of waves in a hyperelastic material. Solutions u corresponding to initial conditions

$$u(x,0) = f(x);\ u_t(x,0) = g(x) \quad \text{for} \quad x \in \mathbb{R}^n \tag{2}$$

may or may not exist "globally", i.e. for all $t \geq 0$. The "life-span" T of a solution is the largest value such that a C^2-solution of (1), (2) exists for $x \in \mathbb{R}^n$, $0 \leq t < T$. Global existence corresponds to $T = \infty$, "blow-up in finite time" to $T < \infty$.

S. Klainerman [1], [2] proved that $T = \infty$ for "sufficiently small" initial data, in case the number n of space dimensions exceeds 3. For initial data of the form

$$u(x,0) = \varepsilon\phi(x), \quad u_t(x,0) = \varepsilon\psi(x) \tag{3}$$

with a constant $\varepsilon > 0$, smallness can be measured conveniently by the size of ε for fixed ϕ,ψ. For $n = 3$ (see Klainerman [2], [3] and John and Klainerman [4]) we only get "almost global" existence of solutions in the sense that $T = T(\varepsilon)$ has a lower bound of the form

$$T > Ae^{B/\varepsilon} \tag{4}$$

with positive constants A,B depending on ϕ,ψ.

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This behavior for higher n constrasts with the case $n = 1$. There it is known (see Lax [17], John [18]) that $T < \infty$ for non-trivial sufficiently small data of compact support, provided the equation is "genuinely nonlinear". More precisely T behaves then like A/ε for small ε. By imbedding, this result for $n = 1$ implies that there exist for any n "large" data* for which $T < \infty$. That actually $T < \infty$ for $n = 3$ and some arbitrarily small data was shown by F. John [5], at least for some equations. An example is the "model" equation

$$\Box u = u_{tt} - \Delta u = F'(u_t)u_{tt} \tag{5}$$

where

$$F(0) = F'(0) = 0; \quad F''(s) > c > 0 \quad \text{for all} \quad s \,. \tag{6}$$

It is shown in [5] that here $T < \infty$ for data (2) of compact support, provided the data satisfy the inequality

$$K = \int_{\mathbb{R}^3} [g(x) - F(g(x))]\, dx > 0 \tag{7}$$

(with $dx = dx_1dx_2dx_3$). More precisely for data of compact support of type (3) for which

$$\int_{\mathbb{R}^3} \psi(x)dx > 0 \tag{8}$$

one has

$$T < A \exp(B/\varepsilon^4) \,. \tag{9}$$

Similar results for other equations with spherical symmetry were obtained by Sideris [6].

These results for $n = 3$ have certain drawbacks. As a consequence of assumption (6) equation (5) becomes elliptic for $u_t > 1/c$. This raises the question if blow-up is just due to this feature, and if it would also occur in equations that are hyperbolic for all arguments**. A second undesirable feature is the inequality restriction (7)

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*Conditions for non-existence of global solutions of systems of conservation laws for sufficiently large data and any n are given by Sideris [19].

**Equation (5) is hyperbolic iff $F'(u_t) < 1$.

imposed on the data, and a third is the fact that the upper bound (9) for T is unrealistically large.

If we restrict ourselves to <u>radial</u> solutions u of (5), (those depending only on $|x|$ and t), the problem becomes essentially one-dimensional and the analysis simplifies, as is shown in [7]. All that matters then for solutions with small initial data is the behavior of $F(s)$ for small s, so that (6) can be replaced by the weaker assumptions

$$F(0) = F'(0) = 0, \quad F''(0) > 0 . \tag{10}$$

For small initial data of type (3) no inequality restriction on $\phi(x),\psi(x)$ is needed, and blow-up for non-trivial data of compact support occurs at a finite time T with an upper bound of the form

$$T < A^* e^{B^*/\varepsilon} \tag{11}$$

for small ε. This bound is optimal in view of the lower bound (4).*

No analogous results for general non-radial solutions u of (5) have been established. The present paper extends the results of [5] to equations (5) that are hyperbolic for all u_t. It proves blow-up in finite time with the optimal bound (11) for T when $n = 3$, but only for data that are subject to a slightly generalized inequality (7). The paper also derives results for $n = 2$ with an inequality**

$$T < A^*/\varepsilon^2 \tag{12}$$

taking the place of (11). The essential difference in the proofs is that here we use plane integrals instead of the spherical means used in [5]. This facilitates a unified treatment of the cases $n = 2,3$. The everywhere hyperbolic character of the differential

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*For analogous results for radial elastic waves see [16].

**The estimate (12) again is optimal, that is T can be shown to have a lower bound $A\varepsilon^{-2}$. This follows by a slight modification of the arguments used in [8] when $n > 3$. One only has to observe that for $n = 2$ the right hand side of formula (98), p. 555 of [8] stays bounded for $T_o = O(\varepsilon^{-2})$.

equation somewhat complicates the argument compared with [5], and makes it necessary to appeal to the energy integral associated with (5).

We assume that our function $F(s)$, is of class $C^3(\mathbb{R})$ and satisfies

$$F(0) = F'(0) = 0; \quad F''(s) > 0, \quad F'(s) \leq q < 1 \quad \text{for all } s \tag{13}$$

so that equation (5) is hyperbolic for all u_t. In what follows n will always have the values 2 or 3. We prescribe initial conditions

$$u(x,0) = f(x) = \varepsilon\phi(x); \quad u_t(x,0) = g(x) = \varepsilon\psi(x) \quad \text{for} \quad x \in \mathbb{R}^n \tag{14}$$

with $\varepsilon > 0$. The data shall have compact support, say

$$f(x) = g(x) = \phi(x) = \psi(x) = 0 \quad \text{for} \quad |x| > R\ . \tag{15}$$

We introduce

$$h(x) = g(x) - F(g(x)) \tag{16}$$

and set

$$K = \int_{\mathbb{R}^n} h(x)dx \tag{17a}$$

$$L(\xi,s) = \int_{x\cdot\xi>-s} [f(x) + (x\cdot\xi + s)h(x)]\, dx \quad \text{for} \quad \xi \in S^{n-1},\ s \in \mathbb{R} \tag{17b}$$

$$\lambda(\xi,s) = \int_{x\cdot\xi>-s} [\phi(x) + (x\cdot\xi + s)\psi(x)]\, dx \quad \text{for} \quad \xi \in S^{n-1},\ s \in \mathbb{R}\ . \tag{17c}$$

Here $dx = dx_1 \ldots dx_n$, $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$, and S^{n-1} denotes the unit sphere in n-space. Under the assumptions (13), (15) we have

$$L(\xi,s) = 0 \quad \text{for} \quad s < -R \tag{18a}$$

$$L(\xi,s) = \int_{\mathbb{R}^n} [f(x) + (x\cdot\xi)h(x)]\, dx + Ks \quad \text{for} \quad s > R \tag{18b}$$

$$L(\xi,s) = \varepsilon\lambda(\xi,s) + O(\varepsilon^2) \quad \text{for fixed } \phi,\psi \text{ and small } \varepsilon\ . \tag{18c}$$

THEOREM. Let $n = 2$ or 3. Let $u(x,t)$ be a C^2-solution of (5) for $x \in \mathbb{R}^n$, $0 < t < T$ with initial data (14) satisfying (15). Then $T < \infty$ if either

$$L(\xi,s) > 0 \quad \text{for some} \quad \xi \in S^{n-1},\ s \in \mathbb{R} \tag{19a}$$

or*

$$K > 0, \quad u \not\equiv 0 . \tag{19b}$$

More precisely, if

$$\lambda(\xi,s) > 0 \quad \text{for some} \quad \xi \in S^{n-1}, \quad s \in \mathbb{R} \tag{19c}$$

then there exist positive A^*, B^* (depending on ϕ,ψ,F) such that for all sufficiently small ε (11) holds when $n = 3$ and (12) holds when $n = 2$.

<u>Corollary</u>. Let $v(x,t)$ for $n = 2,3$ be a nontrivial solution of class C^3 of the nonlinear equation

$$\Box v = v_{tt} - \Delta v = F(v_{tt}) \quad \text{for} \quad x \in \mathbb{R}^N, \quad 0 < t < T \tag{20}$$

where F satisfies (13). Let v have initial values $v(x,0)$ and $v_t(x,0)$ of compact support. Then $T < \infty$.

<u>Proof of the Corollary</u>. The function $u = v_t$ is a C^2-solution of (5) for which $u(x,0) = v_t(x,0)$ and

$$h(x) = u_t(x,0) - F(u_t(x,0)) = v_{tt}(x,0) - F(v_{tt}(x,0)) = \Delta v(x,0)$$

have compact support. Then $u_t(x,0)$ also has compact support, since by (13)

$$|s - F(s)| = \left|\int_0^s (1 - F'(z))dz\right| \geq (1 - q)|s| \neq 0 \quad \text{for} \quad s \neq 0 .$$

Moreover here

$$K = \int_{\mathbb{R}^n} h(x)\, dx = \int_{\mathbb{R}^n} \Delta v(x,0)\, dx = 0 .$$

Applying the Theorem for $T = \infty$ yields $u(x,t) = v_t(x,t) \equiv 0$. Then $v(x,t) = v(x,0)$ and by (20) $\Delta v(x,0) = 0$, which implies $v(x,0) = 0$, since $v(x,0)$ has compact support. It follows that $v(x,t) \equiv 0$.

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*Observe that $K > 0$ implies by (18b) that (19a) holds for all sufficiently large s.

<u>Proof of the Theorem</u>. The proof uses the common type of argument that might be called "method of moments". (See e.g. references [10],[11],[12],[13],[14],[15],[19].) A differential inequality is established for a certain "moment" (a functional in integral form formed from the solution). On the basis of this inequality the moment is shown to grow with time in a manner incompatible with continued existence. By this method one proves non-existence of a global solution, without, however, gaining any insight into the process of singularity formation constituting blow-up. The actual blow-up involving possibly only higher derivatives, quite likely, takes place some time before the moments in question show any drastic behavior. The method of moments then just confirms that the solution (in the strict sense) has disintegrated after a finite time, without establishing the cause of death.

First of all $u(x,t)$ is of compact support in x. More precisely for data satisfying (15) we have

$$u(x,t) = 0 \quad \text{for} \quad |x| > R + t \tag{21}$$

(see [5], p. 49). Introducing

$$v(x,t) = \int_0^t u(x,s)\, ds \quad \text{for} \quad x \in \mathbb{R}^n, \quad 0 < t < T \tag{22}$$

we have $v_t = u$ and

$$v(x,0) = 0, \quad v_t(x,0) = f(x)$$

$$v(x,t) = 0 \quad \text{for} \quad |x| > R + t \tag{23}$$

$$\Box v(x,t) = F(v_{tt}(x,t)) + h(x) \quad \text{for} \quad x \in \mathbb{R}^n, \quad 0 < t < T \tag{24}$$

with $h(x)$ defined by (16). We associate with the function $v(x,t)$ the "plane integral"

$$v^*(r,t) = \int_{x_1=r} v(x,t)\, dS \quad \text{for} \quad r \in \mathbb{R}, \quad 0 < t < T \tag{25}$$

where $dS = dx_1 \ldots dx_n/dx_1$, that is $dS = dx_2$ when $n = 2$ and $dS = dx_2 dx_3$ when $n = 3$. Then

$$v^*(r,t) = 0 \quad \text{for} \quad |r| > R + t \tag{26a}$$

$$v^*(r,0) = 0$$

$$v_t^*(r,0) = f^*(r) = \int_{x_1=r} f(x)\, dS \tag{26b}$$

$$v_{tt}^*(r,t) - v_{rr}^*(r,t) = F^*(r,t) + h^*(r) \tag{26c}$$

where we define

$$F^*(r,t) = \int_{x_1=r} F(v_{tt}(x,t))\, dS \tag{26d}$$

$$h^*(r) = \int_{x_1=r} h(x)\, dS\ . \tag{26e}$$

It follows that

$$v^*(r,t) = v_o^*(r,t) + \frac{1}{2}\int_{T_{r,t}} F^*(\rho,\tau)\, d\rho d\tau \quad \text{for} \quad r \in R,\quad 0 < t < T \tag{27a}$$

where $T_{r,t}$ is the "characteristic" triangle with vertices $(r,t),(r - t,0),(r + t,0)$ and

$$v_o^*(r,t) = \frac{1}{2}\int_{r-t}^{r+t} [f^*(\rho) + (t - |r - \rho|)h^*(\rho)]\, d\rho\ . \tag{27b}$$

Since by (15), (16)

$$f^*(\rho) = h^*(\rho) = 0 \quad \text{for} \quad |\rho| > R \tag{28}$$

we have

$$v_o^*(r,t) = \frac{1}{2}\int_{r-t}^{R} [f^*(\rho) + (t - r + \rho)h^*(\rho)]\, d\rho = \frac{1}{2} M(t - r) \tag{29}$$

for $r > R$, $t > 0$, where in the notation of (17b)

$$M(z) = \int_{-z}^{R} [f^*(\rho) + (z + \rho)h^*(\rho)]\, d\rho$$

$$= \int_{x_1 > -z} [f(x) + (z + x_1)h(x)]\, dx = L(\xi^1, z) \qquad (30)$$

with ξ^1 denoting the unit vector in the x_1-direction:

$$\xi^1 = (1,0,\ldots,0)\ . \qquad (31)$$

Our assumptions (15) on F imply that

$$F(z) > 0 \quad \text{for} \quad z \neq 0 \qquad (32a)$$

and that there exist positive constants a,b such that

$$F(z) \geq az^2 \quad \text{for} \quad |z| \leq b\ . \qquad (32b)$$

Then by (15)

$$\frac{F(z)}{z} \geq \frac{F(b)}{b} \geq ab \quad \text{for} \quad z \geq b \qquad (32c)$$

$$ab \leq q < 1\ . \qquad (32d)$$

Since $F(z)$ is convex and $v_{tt}(x,t)$ has its support in $|x| \leq R + t$ we have from Jensen's inequality applied to (26d)

$$\frac{F^*(r,t)}{c(r,t)} \geq F\left(\frac{1}{c(r,t)} \int_{x_1=r} v_{tt}(x,t)\, dS\right)$$

$$= F\left(\frac{v^*_{tt}(r,t)}{c(r,t)}\right) \geq 0 \qquad (33a)$$

for $r < t + R$. Here

$$c(r,t) = \int_{\substack{x_1=r \\ |x|<t+R}} dS = \gamma((t + R)^2 - r^2) \qquad (33b)$$

with $\gamma(z)$ defined by

$$\gamma(z) = 0 \quad \text{for} \quad z < 0,\ \gamma(z) = 2z^{1/2} \quad \text{for} \quad z > 0 \quad \text{when} \quad n = 2 \qquad (33c)$$

$$\gamma(z) = 0 \quad \text{for} \quad z < 0,\ \gamma(z) = \pi z \quad \text{for} \quad z > 0 \quad \text{when} \quad n = 3\ . \qquad (33d)$$

Blow-up will be established by deriving an integral inequality for $v^*(r,t)$ along lines $t - r = \text{const.} = z$. In what follows let z be a fixed number with

$$z > -R\ . \qquad (34a)$$

Define

$$P(r) = v^*(r,z+r) \quad \text{for} \quad R \le r < T - z \; . \tag{34b}$$

By (27a), (29)

$$P(r) = M(z) + \frac{1}{2}\int_{T_{r,z+r}} F^*(\rho,\tau)\, d\rho d\tau \quad \text{for} \quad R \le r < T - z \; . \tag{34c}$$

For $R \le r_1 \le r < T - z$ we have $T_{r_1,z+r_1} \subset T_{r,z+r}$ and

$$P(r) = P(r_1) + \frac{1}{2}\int_{\Gamma} F^*(\rho,\tau)\, d\rho d\tau \tag{34d}$$

with $\Gamma = T_{r,z+r} \setminus T_{r_1,z+r_1}$. Set

$$r_2 = z + 2r_1 \; . \tag{34e}$$

Then for $r_2 < r$ the region Γ contains the parallelogram

$$r_2 < \rho < r, \quad \rho - R < \tau < \rho + z \tag{34f}$$

and it follows from (34d), (33a) that

$$P(r) \ge P(r_1) + \frac{1}{2}\int_{r_2}^{r} d\rho \int_{\rho-R}^{\rho+z} c(\rho,\tau) F\left(\frac{v^*_{tt}(\rho,\tau)}{c(\rho,\tau)}\right) d\tau \tag{34g}$$

for $r_2 < r < T - z$.

Since by (26a) $v^*(\rho,\tau) = 0$ for $\tau < \rho - R$, we have for $R < \rho < T - z$

$$P(\rho) = v^*(\rho,z+\rho) = \int_{\rho-R}^{\rho+z} (\rho + z - \tau) v^*_{tt}(\rho,\tau)\, d\tau \; . \tag{35a}$$

Then by Jensen's inequality and the convexity of F

$$C(\rho) F\left(\frac{P(\rho)}{C(\rho)}\right) \le \int_{\rho-R}^{\rho+z} (\rho + z - \tau) c(\rho,\tau) F\left(\frac{v^*_{tt}(\rho,\tau)}{c(\rho,\tau)}\right) d\tau \tag{35b}$$

where

$$C(\rho) = \int_{\rho-R}^{\rho+z} (\rho + z - \tau)c(\rho,\tau)\, d\tau \ , \tag{35c}$$

(35b) yields

$$C(\rho)F\Big(\frac{P(\rho)}{C(\rho)}\Big) \leq (z + R) \int_{\rho-R}^{\rho+z} c(\rho,\tau)F\Big(\frac{v^*_{tt}(\rho,\tau)}{c(\rho,\tau)}\Big)\, d\tau \ . \tag{35d}$$

Substituting into (34g) gives the desired inequality for P:

$$P(r) \geq P(r_1) + \frac{1}{2}\int_{r_2}^{r} \frac{C(\rho)}{z + R} F\Big(\frac{P(\rho)}{C(\rho)}\Big)\, d\rho \quad \text{for} \quad r_2 < r < T - r \ . \tag{35e}$$

<u>Lemma 1</u>. Let there exist r_1, t_1, k with

$$r_1 \geq R, \quad 0 \leq t_1 < T, \quad v^*(r_1,t_1) = k > 0 \ . \tag{36}$$

Then $T < \infty$.

<u>Proof of Lemma 1</u>. Set $z = t_1 - r_1$. Then $z > -R$ by (26a). Define P, r_2 as in (34b), (34e). Then

$$P(r_1) = k > 0 \ . \tag{37a}$$

We compare $P(r)$ with the solution $p(r)$ of the integral equation

$$p(r) = P(r_1) + \frac{1}{2}\int_{r_2}^{r} \frac{C(\rho)}{z + R} F\Big(\frac{p(\rho)}{C(\rho)}\Big)\, d\rho \quad \text{for} \quad r \geq r_2 \tag{37b}$$

that is with the solution fo the differential equation problem

$$p'(r) = \frac{C(r)}{2(z + R)} F\Big(\frac{p(r)}{C(r)}\Big), \qquad p(r_2) = P(r_1) \ . \tag{37c}$$

Since $p(r) > 0$ by (37a), (32a), and F is increasing for positive arguments by (13), we have

$$P(r) \geq p(r) \quad \text{for} \quad r_2 < r < T - z \tag{37d}$$

by Gronwall's lemma.

The inequalities (32b,c) furnish different lower bounds for $F(p(\rho)/C(\rho))$ as $p/C > b$ or $< b$. Let $\rho \geq r_2$ be a value for which

$$\frac{p(\rho)}{C(\rho)} > b . \tag{38a}$$

Then by (35c), (32c)

$$\frac{d\,\frac{p(\rho)}{C(\rho)}}{d\rho} = \frac{1}{2(z + R)} F\left(\frac{p(\rho)}{C(\rho)}\right) - \frac{p(\rho)C'(\rho)}{C^2(\rho)}$$

$$> \frac{p(\rho)}{C(\rho)} \left(\frac{ab}{2(z + R)} - \frac{C'(\rho)}{C(\rho)}\right) .$$

Here by (35c), (33b)

$$C(\rho) = \int_{\rho-R}^{\rho+z} (\rho + z - \tau)\gamma((\tau + R)^2 - \rho^2)\, d\tau$$

$$= \int_0^{z+R} (z + R - \sigma)\gamma(\sigma(\sigma + 2\rho))\, d\sigma \tag{38b}$$

$$C'(\rho) = \int_0^{z+R} (z + R - \sigma)\gamma(\sigma(\sigma + 2\rho)) \frac{2\sigma\gamma'(\sigma(\sigma + 2\rho))}{\gamma(\sigma(\sigma + 2\rho))}\, d\sigma < \frac{C(\rho)}{\rho}$$

since by (33c,d)

$$\frac{\gamma'(\sigma(\sigma + 2\rho))}{\gamma(\sigma(\sigma + 2\rho))} = \frac{n - 1}{2\sigma(\sigma + 2\rho)} < \frac{1}{2\sigma\rho} .$$

Thus

$$\frac{dp(\rho)/C(\rho)}{d\rho} > 0 \quad \text{for} \quad \rho > \frac{2(z + R)}{ab} .$$

Set

$$r_3 = \text{Max}\left(r_2, \frac{2(z + R)}{ab}\right) . \tag{38c}$$

Then

$$\frac{p(\rho)}{C(\rho)} > b, \quad \rho > r_3 \tag{38d}$$

implies that

$$\frac{p(r)}{C(r)} > b \quad \text{for} \quad r > \rho . \tag{38e}$$

Let now r be such that

$$r_3 \leq r < T - z, \quad p(\rho) \leq bC(\rho) \quad \text{for} \quad r_3 < \rho < r . \tag{39a}$$

Then by (32b), (37c), (39a)

$$p'(\rho) > \frac{ap^2(\rho)}{2(z + R)C(\rho)} \quad \text{for} \quad r_3 < \rho < r \tag{39b}$$

$$\frac{1}{k} = \frac{1}{p(r_2)} > \frac{1}{p(r_3)} > \frac{1}{p(r_3)} - \frac{1}{p(r)} > \frac{a}{2(z + R)} \int_{r_3}^{r} \frac{d\rho}{C(\rho)} .$$

Here by (38b), (33c,d)

$$C(\rho) < (z + R)^2\gamma((z + R)(z + R + 2\rho)) < \pi(z + R)^{(n+3)/2}(3\rho)^{(n-1)/2} \tag{39c}$$

since by (38c), (32d)

$$z + R + 2\rho < \frac{1}{2}\, abr_3 + 2\rho < 3\rho \quad \text{for} \quad r_3 < \rho ,$$

$$\int_{r_3}^{r} \frac{d\rho}{C(\rho)} > \frac{1}{3\pi} (z + R)^{-(n+3)/2} \int_{r_3}^{r} \rho^{(1-n)/2} d\rho . \tag{39d}$$

We define r_4 by

$$\frac{1}{k} = \frac{a}{2(z + R)} \int_{r_3}^{r_4} \frac{d\rho}{C(\rho)} . \tag{39e}$$

Then

$$P(r) > p(r) > bC(r) \quad \text{for} \quad r_4 < r < T - z . \tag{39f}$$

We have by (35a)

$$P(r) = \int_{x_1=r} dS \int_{r-R}^{r+z} (z + r - t)v_{tt}(x,t)\, dt .$$

Hence

$$P^2(r) < J \int_{\substack{x_1=r \\ r-R<t<r+z}} v_{tt}^2(x,t)\, dSdt$$

where

$$J = \int\limits_{\substack{x_1=r,\,|x|<t+R \\ r-R<t<r+z}} (z + r - t)^2 \, dSdt$$

$$< (z + R) \int_{r-R}^{r+z} (z + r - t)c(r,t) \, dt = (z + R)C(r) \; .$$

It follows from (39f) that for $r_4 < r < T - z$

$$b^2C(r) < \frac{P^2(r)}{C(r)} < (z + R) \int\limits_{\substack{x_1=r \\ r-R<t<r+z}} v_{tt}^2(x,t) \, dSdt \; . \qquad (39g)$$

Here by (38b)

$$C(r) > \int_0^{z+R} (z + R - \sigma)\gamma(2r\sigma) \, d\sigma$$

$$> 2 \int_0^{z+R} (z + R - \sigma)(2\sigma)^{(n-1)/2} r^{(n-1)/2} \, d\sigma$$

$$> \frac{2}{3} (z + R)^{(n+3)/2} r^{(n-1)/2} \; . \qquad (39h)$$

Let now ρ be a number with $r_4 < \rho < T - 2z - R$ so that

$$r_4 < r < T - z \quad \text{for} \quad \rho < r < \rho + z + R \; .$$

Integrating (39g) with respect to r from ρ to $\rho + z + R$ and using (39h) we find that

$$\frac{2}{3} b^2(z + R)^{(n+3)/2}\rho^{(n-2)/2} < \int\limits_{\substack{\rho<x_1<\rho+z+R \\ x_1-R<t<x_1+z}} v_{tt}^2(x,t) \, dxdt$$

$$< \int\limits_{\rho-R<t<\rho+2z+R} v_{tt}^2(x,t) \, dxdt \; . \qquad (39i)$$

Introducing

$$G(s) = 2 \int_0^s zF'(z) \, dz \qquad (40a)$$

we have associated with (5) the "conservation of energy" relation

$$E = \frac{1}{2} \int [u_t^2 - G(u_t) + (\nabla u)^2]\, dx = \text{const.}$$
$$= \frac{1}{2} \int [g^2 - G(g) + (\nabla f)^2]\, dx \,. \tag{40b}$$

Here, because of assumptions (13)

$$G(s) \leq qs^2 \leq s^2 \quad \text{for all} \quad s \in R \,. \tag{40c}$$

Thus

$$\int v_{tt}^2(x,t)\, dx = \int u_t^2(x,t)\, dx \leq \frac{2E}{1-q} \quad \text{for} \quad 0 \leq t < T \,. \tag{40d}$$

It follows from (39i) that

$$\frac{1-q}{6} b^2(z+R)^{(n+1)/2} \rho^{(n-1)/2} \leq E \quad \text{for} \quad r_4 < \rho < T - 2z - R \,. \tag{40c}$$

Consequently either

$$T - 2z - R < r_4 \tag{41a}$$

or

$$\frac{1-q}{6} b^2(z+R)^{(n+1)/2}(T - 2z - R)^{(n-1)/2} \leq E \tag{41b}$$

holds. In either case $T < \infty$, proving the lemma.

<u>Lemma 2</u>. Let there exist $\xi \in S^{n-1}$, $s \in R$ such that

$$k = 2L(\xi,s) > 0 \,. \tag{42a}$$

Then $T < \infty$. More precisely there exist positive constants α,β only depending on the choice of F such that

$$T < \alpha(s+R)\, \mathrm{Max}\left[\exp\left(\frac{\beta(s+R)^4}{k}\right), \frac{E}{(s+R)^3}\right] \quad \text{when} \quad n = 3 \tag{42b}$$

$$T < \alpha(s+R)\, \mathrm{Max}\left[1 + \frac{(s+R)^6}{k^2}, \frac{E^2}{(s+R)^4}\right] \quad \text{when} \quad n = 2 \,. \tag{42c}$$

<u>Corollary</u>. For initial data of type (14) and $\lambda(\xi,s) > 0$ there exist positive constants A^*,B^* depending on ϕ,ψ,F such that for all sufficiently small $\varepsilon > 0$ relation (11) holds when $n = 3$ and (12) when $n = 2$.

Proof of the Corollary. For fixed ϕ,ψ we have using (13) that $k = L(\xi,s) = \varepsilon\lambda(\xi,s) + O(\varepsilon^2)$, $E = O(\varepsilon^2)$ for small $\varepsilon > 0$. Then (11), (12) follow immediately from (42b,c).

Proof of Lemma 2. By (18a) $s > -R$. Since equation (5) is invariant under rigid motions, we can bring about that the ξ in (42a) is the unit vector ξ^1 defined in (31). Then (29), (30) yield

$$v_0^*(R,s + R) = \frac{1}{2} M(s) = k > 0 . \tag{43a}$$

Thus (36) holds with

$$r_1 = R, \quad t_1 = s + R . \tag{43b}$$

It follows from (34e), (38c), (32d) that here

$$r_2 = s + 2R, \quad r_3 = \frac{2(s + R)}{ab} . \tag{43c}$$

Using the estimate (39c) for $C(\rho)$ we find from (39e) that

$$r_4 < \frac{2(s + R)}{ab} \exp\left[\frac{6\pi(s + R)^4}{ak}\right] \quad \text{when} \quad n = 3 \tag{43d}$$

$$r_4 < \frac{4(s + R)}{ab} + \frac{6\pi^2(s + R)^7}{a^2k^2} \quad \text{when} \quad n = 2 . \tag{43e}$$

Thus (41a) implies that

$$T < \frac{4(s + R)}{ab} \exp\left[\frac{6\pi(s + R)^4}{ak}\right] \quad \text{when} \quad n = 3 \tag{43f}$$

$$T < \frac{6(s + R)}{ab} + \frac{6\pi^2(s + R)^7}{a^2k^2} \quad \text{when} \quad n = 2 . \tag{43g}$$

On the other hand (41b) leads to

$$T < 2(s + R) + \frac{6E}{(1 - q)b^2(s + R)^2} \quad \text{when} \quad n = 3 \tag{44a}$$

$$T < 2(s + R) + \frac{36E^2}{(1 - q)^2b^4(s + R)^3} \quad \text{when} \quad n = 2 . \tag{44b}$$

This establishes (42b,c) and proves the lemma.

<u>Lemma 3</u>. $K = 0$ implies that either $T < \infty$ or $u \equiv 0$.

<u>Corollary</u>. Completion of the proof of the Theorem.

<u>Proof of Lemma 3</u>. Assume that $T = \infty$. For any fixed $z > -R$ define $P(r)$ by (34b) for $r > r_1 = R$. Then

$$P(r) < 0 \tag{45a}$$

by Lemma 1. The representation (34c) for P combined with $F^* > 0$ shows that $P(r)$ is non-decreasing in r for $r > R$. Hence

$$\delta = \lim_{r\to\infty} \frac{1}{2} \int_{T_{r,z+r}} F^*(\rho,\tau)\, d\rho d\tau = -M(z) + \lim_{r\to\infty} P(r) \tag{45b}$$

exists, and

$$0 < \delta < -M(z) . \tag{45c}$$

If here

$$\delta + M(z) = -m < 0$$

it would follow that

$$P(\rho) < -m < 0 \quad \text{for} \quad \rho > R . \tag{45d}$$

There exists a ρ^* such that

$$\rho^* > r_2 = z + 2R, \quad -b < \frac{-m}{C(\rho)} < 0 \quad \text{for} \quad \rho > \rho^* .$$

By (32b) then

$$F\Big(\frac{P(\rho)}{C(\rho)}\Big) > F\Big(\frac{-m}{C(\rho)}\Big) > \frac{am^2}{C^2(\rho)} \quad \text{for} \quad \rho > \rho^* . \tag{45e}$$

Using (35e), (39c) we find that

$$P(r) > P(R) + \frac{1}{2}\int_{\rho^*}^{r} \frac{am^2}{C(\rho)} \frac{d\rho}{z+R}$$

$$> P(R) + \frac{3^{(1-n)/2}am^2}{2\pi} (z+R)^{-(n+5)/2} \int_{\rho^*}^{r} \rho^{(1-n)/2}\, d\rho .$$

But this implies $P(r) > 0$ for all sufficiently large r, contrary to (45a).

Hence*

$$\delta = \frac{1}{2} \int_{0<\tau<Z+\rho} F^*(\rho,\tau)\, d\rho d\tau = -M(z) \quad \text{for} \quad z > -R \; .$$

It follows from (30), (18b) and $K = 0$ that

$$\int_{z_1+\rho<\tau<z_2+\rho} F^*(\rho,\tau)\, d\rho d\tau = 0 \quad \text{for} \quad R < z_1 < z_2 \; .$$

Consequently

$$F^*(\rho,\tau) = 0 \quad \text{for} \quad \tau > \rho + R, \quad \tau > 0$$

and thus by (26d), (32a)

$$u_t(x,t) = v_{tt}(x,t) = 0 \quad \text{for} \quad t > x_1 + R, \quad t > 0 \; . \tag{46a}$$

Using the spherical symmetry of equation (5) and of K we deduce from (46a) that more generally

$$u_t(x,t) = 0 \quad \text{for} \quad t > x \cdot \xi + R \quad \text{with any} \quad \xi \in S^{n-1}, \quad t > 0 \; .$$

But then

$$u_t(x,t) = 0 \quad \text{for} \quad t > R - |x|, \quad t > 0$$

and in particular

$$u_t(x,t) = 0 \quad \text{for} \quad t > R, \quad \text{and all} \quad x \in \mathbb{R}^n.$$

It follows from (5) that

$$\Delta u(x,t) = 0 \quad \text{for} \quad t > R, \quad x \in \mathbb{R}^n$$

and then from (21) that

$$u(x,t) = 0 \quad \text{for} \quad t > R, \quad x \in \mathbb{R}^n \; .$$

The uniqueness theorem for equation (5), (see [5], p. 49) then yields that also

$$u(x,t) = 0 \quad \text{for} \quad 0 < t < T, \quad x \in \mathbb{R}^n \; ,$$

completing the proof of Lemma 3.

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*This identity holds whenever $T = \infty$, regardless of the value of K.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with solutions $u(x_1,\ldots,x_n,t) = u(x,t)$ of nonlinear partial differential equations of the form $\Box u = u_{tt} - \Delta u = F'(u_t)u_{tt}$ for prescribed initial values $u(x,0) = \varepsilon\phi(x)$, $u_t(x,0) = \varepsilon\psi(x)$ of compact support. Here the assumptions $F(0) = F'(0) = 0$, $F'' > 0$, $F' \leq q < 1$ ensure

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20. ABSTRACT - cont'd.

hyperbolicity of the equation. It is known that for $n > 3$ smooth solutions exist for $x \in \mathbb{R}^n$ and all $t > 0$, provided ε is sufficiently small. It is shown here that no such "global" solutions need to exist for arbitrarily small ε, when $n = 2$ or 3. More precisely, if ϕ and ψ satisfy certain inequalities there exist positive constants A,B such that no classical solution exists for $t > Ae^{B/\varepsilon}$ when $n = 3$ and for $t > A/\varepsilon^2$ when $n = 2$. These upper bounds for the "life span" of u are optimal. For the proof one shows that certain plane integrals of u become larger for large t than is consistent with the value of the total energy derived from the initial data.

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